Real Smooth Points

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Definitions

- Given \( f_1, \ldots, f_s, g_1, \ldots, g_t \in \mathbb{R}[x] \)

\[
S = \{ x \in \mathbb{R}^n \mid f_1(x) = \cdots = f_s(x) = 0, g_1(x) > 0, \ldots, g_t(x) > 0 \}.
\]

is an atomic semi-algebraic set. If \( t = 0 \), \( S \) is a real algebraic set.

- A point \( z \in S \) is smooth (or nonsingular) in \( S \) if \( z \) is smooth in the algebraic set

\[
V(f_1, \ldots, f_s) = \{ x \in \mathbb{C}^n : f_1(x) = \cdots = f_s(x) = 0 \},
\]

i.e. if there exists a unique irreducible component \( V \subset V(f_1, \ldots, f_s) \) containing \( z \) such that

\[
\dim T_z(V) = \dim V
\]

where \( T_z(V) \) is the tangent space of \( V \) at \( z \).

- We denote by \( \text{Sing}(S) \) the set of singular (or non-smooth) points in \( S \).
The Problem

Problem

Given $S$, an atomic semi-algebraic set.

Find a smooth point in each connected component of $S$.

Applications:

- **Kuramoto model**: a dynamical system to model synchronization amongst $n$ coupled oscillators. Computational proof for max number of equilibrium for $n = 4$.

- **Real Dimension**: Try to close the complexity gap between real and complex case.

  Let $V = V(f_1, \ldots, f_s) \subset \mathbb{C}^n$, $d := \max_i \deg f_i$ and $V_{\mathbb{R}} = V \cap \mathbb{R}^n$.

  Best known algorithms:
  - Find $\dim_{\mathbb{C}} V$: $d^{O(n)}$ worst case running time.
  - Find $r := \dim_{\mathbb{R}} V_{\mathbb{R}}$: $d^{O(r(n-r))}$ worst case running time.

**Note**: Every atomic semialgebraic set is a projection of a real algebraic set.

⇒ We can always rewrite our semialgebraic set as algebraic by adding variables, e.g.

\[
g(x) \leq 0 \iff g(x) + \gamma^2 = 0
\]

\[
g(x) < 0 \iff \gamma^2 g(x) + 1 = 0
\]
Previous Work

Finding smooth points in each connected component of $V_\mathbb{R}$

- **Cell decomposition based on sign conditions**
  - Collins’ CAD (1975)
  - Basu, Pollack, Roy (2006) - Chapter 13

- **Critical points of distance function and polar varieties**
  
  *Only guaranteed to work for smooth varieties*
  
  - Seidenberg (1954)
  - Roullier, Roy, Safey El Din (2000)
  - Aubry, Rouillier, Safey El Din (2002)
  - Safey El Din and Schost (2003)
  - Faugere et al. (2008)
  - Hauenstein (2012)
  - Wu and Reid (2013)
  - Draisma et al. (2016)
  - Safey El Din and Spaenlehauer (2016)
  - Safey El Din, Yang, Zhi (2018)
  - Elliott, Giesbrecht and Schost (2020)

Computing Real Dimension

- Collins (1975)
- Koiran (1999)
- Vorobjov (1999)
- Basu, Pollack, Roy (2006)
- Safey El Din and Tsigaridas (2013)
- Bannwarth and Safey El Din (2015)
Theorem [Harris, Hauenstein, Sz.]

Let \( f_1, \ldots, f_s \in \mathbb{R}[x_1, \ldots, x_n] \) and assume that \( V := V(f_1, \ldots, f_s) \subset \mathbb{C}^n \) is equidimensional of dimension \( n - s \). Suppose that \( g \in \mathbb{R}[x_1, \ldots, x_n] \) satisfies the following conditions:

1. \( \text{Sing}(V) \cap \mathbb{R}^n \subset V(g) \);
2. \( \dim (V \cap V(g)) < n - s \).

Then the set of points where \( g \) restricted to \( V \cap \mathbb{R}^n \) attains its extreme values intersects each bounded connected component of \( (V \setminus \text{Sing}(V)) \cap \mathbb{R}^n \).

Algorithmically: We find the critical points of \( g \) in \( V \) by solving

\[
L := \left\{ \frac{\partial g}{\partial x_i} + \sum_{t=1}^{s} \lambda_t \frac{\partial f_t}{\partial x_i} : i = 1, \ldots, n \right\} \cup \{f_1, \ldots, f_s\}.
\]

in the variables \( x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_s \).

Note: The challenge is finding a low degree \( g \) satisfying the above two conditions. I will get back to how to compute such a \( g \) at the end of talk.
Example: Mork-Piene Curve

Real plane curve (Mork-Piene 2008): critical points of the distance function from any point in $\mathbb{R}^2$ will not contain smooth points on all four connected components:

\[ f_1 = (x^2 + y^2 - 1)((x - 4)^2 + (y - 2)^2 - 1) \]
\[ f_2 = \left( y - \frac{1}{2} \right) \left( y + \frac{1}{2} \right) \left( x - \frac{7}{2} \right) \left( x - \frac{9}{2} \right) \]
\[ F = f_1^2 + \frac{1}{100} f_2^3 \]
\[ g = (4x^2 - 3)(4y^2 - 1)(4x^2 - 32x + 63)(4y^2 - 16y + 13) \]
Application to Kuramoto Model for $n = 4$

Kuramoto model (1975): a dynamical system to model synchronization amongst $n$ coupled oscillators.

Open problem: Find the maximum number of equilibria for $n \geq 4$.

Polynomial system: Compute max number isolated real solutions of $F = 0$ as $\omega \in \mathbb{R}^3$ for

$$F(s, c; \omega) = \left\{ \omega_i - \frac{1}{4} \sum_{j=1}^{4} (s_i c_j - s_j c_i), s_i^2 + c_i^2 - 1, s_4, c_4 - 1, \text{ for } i = 1, 2, 3 \right\}.$$  

Our approach:

1. Compute the discriminant $D(\omega)$ of the system $F$: $\deg D(\omega) = 48$.
2. Compute sample points in each bounded connected components of $\mathbb{R}^3 \setminus V(D(\omega))$ computing the critical points of $D(\omega)$, i.e. solve the system:

$$\nabla D(\omega) = 0 \quad D(\omega) \neq 0.$$  

Note: Bezout bound for this system is $47^3 > 100k$.

3. For each real sample points $\tilde{\omega} \in \mathbb{R}^3$ compute the real solutions of $F(s, c; \tilde{\omega})$.
4. Certify the solutions using alphaCertified (Hauenstein, Sottile 2012)
Kuramoto model and symmetries

Add the polynomial $\omega_4 - \frac{1}{4} \sum_{j=1}^{4} (s_4 c_j - s_j c_4)$ to get a system $\bar{F}(s, c; \bar{\omega})$ with $\bar{\omega} \in \mathbb{R}^4$.

Note: $\omega_1 + \omega_2 + \omega_3 + \omega_4 = 0$.

→ Discriminant $\bar{D}(\bar{\omega})$ of $\bar{F}$ is a symmetric polynomial
→ $\bar{D}(\bar{\omega}) = H(e)$ with $e = (e_1, \ldots, e_4)$ elementary symmetric polynomials
→ $\nabla_{\bar{\omega}} \bar{D}(\bar{\omega}) = M \cdot \nabla_e H(e)$ where $\det(M) = \prod_{1 \leq i < j \leq 4} (\omega_i - \omega_j)$

- $M$ non-singular: 105 solutions, orbit size 48 → 5040 solutions
- $M$ singular: 4 subsystems, further symmetries: 1292 solutions

Theorem [Harris, Hauenstein, Sz.]
The maximum number of equilibria for the Kuramoto model with $n = 4$ oscillators is 10.

\[ \omega_3 = \omega_4 = -\left(\omega_1 + \omega_2\right)/2 \]
\[ \omega_3 = \omega_4 = -\left(\omega_1 + \omega_2\right)/2 \]
\[ \omega_2 = -\omega_1, \omega_4 = -\omega_3 \]
\[ \omega_2 = -\omega_1, \omega_4 = -\omega_3 \]

(a) one slice  (b) zoomed in  (c) other slice  (d) zoomed in

Figure 1: Bounded connected regions and critical points, Kuramoto model, $n = 4$. 
Computation of Real Dimension

**Theorem [Marshall 2008]**

For $V \subset \mathbb{C}^n$ an irreducible algebraic set,

$$\dim_{\mathbb{R}} V \cap \mathbb{R}^n = \dim_{\mathbb{C}} V$$

if and only if there exists a smooth $z \in V \cap \mathbb{R}^n$.

**Main idea of a Real Dimension Algorithm:**

- If there exists a smooth $z \in V \cap \mathbb{R}^n$ then $\dim_{\mathbb{R}}(V \cap \mathbb{R}^n) = \dim_{\mathbb{C}}(V)$.
- If not, $V \cap \mathbb{R}^n \subseteq \text{Sing}(V)$ and $\dim_{\mathbb{R}}(V \cap \mathbb{R}^n) < \dim_{\mathbb{C}}(V)$.
- Lower the complex dimension without losing real points, i.e. find $V' \subset V$ algebraic set such that

  $$\dim_{\mathbb{C}} V' = \dim V - 1 \text{ and } V' \cap \mathbb{R}^n = V \cap \mathbb{R}^n.$$ 

  Next slide: $V'$ is the limit of a perturbed polar varieties.

- Iterate using $V'$ instead of $V$. 
Limits of Perturbed Polar Varieties

\[ f := \sum_{i=1}^{s} f_i^2 \in \mathbb{R}[x_1, \ldots, x_n], \pi_i : \mathbb{C}^n \to C^i \text{ projection, } \varepsilon \text{ infinitesimal}, \ V_\varepsilon = V(f - \varepsilon). \]

For \( i = 1, \ldots n \) the \((i - 1)\)-th polar variety of \( V_\varepsilon \) is defined as

\[ \text{crit}(V_\varepsilon, \pi_i) := V \left( f - \varepsilon, \frac{\partial f}{\partial x_{i+1}}, \ldots, \frac{\partial f}{\partial x_n} \right) \subset \mathbb{C}^n. \]

**Theorem [Safey El Din, Tsigaridas 2013]**

After a generic change of variables, \( \text{crit}(V_\varepsilon, \pi_i) \) are either empty or smooth and equidimensional of dimension \( i - 1 \) for \( i = 1, \ldots, n. \)

Furthermore, if \( \dim_{\mathbb{R}}(V(f) \cap \mathbb{R}^n) < i \) then

\[ V(f) \cap \mathbb{R}^n = \lim_{\varepsilon \to 0} \text{crit}(V_\varepsilon, \pi_i) \cap \mathbb{R}^n. \]
Theorem [Harris, Hauenstein, Sz.]

Let \( f_1, \ldots, f_s \in \mathbb{R}[x] \), fix \( a = (a_1, \ldots, a_s) \in \mathbb{R}^s \) such that for all sufficiently small \( \varepsilon > 0 \)
\[ V_\varepsilon := V(f_1 - a_1\varepsilon, \ldots, f_s - a_s\varepsilon) \subset \mathbb{C}^n \] is smooth and equidimensional of dimension \( n - s \).
Let \( V = \lim_{\varepsilon \to 0} V_\varepsilon \).
Let \( g \in \mathbb{R}[x] \) such that

- \( \text{Sing}(V) \cap \mathbb{R}^n \subset V(g) \) and
- \( \dim (V \cap V(g)) < n - s \).

Let \( C_\varepsilon \subset \mathbb{C}^n \) be the set of critical points of \( g \) on \( V_\varepsilon \). Then \( C_\varepsilon \) is finite.
Furthermore, let
\[ S := \left( \lim_{\varepsilon \to 0} C_\varepsilon \right) \setminus V(g) \cap \mathbb{R}^n. \]

If \( S = \emptyset \), then \( V \cap \mathbb{R}^n \) has no bounded connected components of dimension \( n - s \). If \( S \neq \emptyset \), then \( V \cap \mathbb{R}^n \) has some connected components (possibly unbounded) of dimension \( n - s \), and \( S \) contains smooth points in each of these components.

Note: One can find such \( g \) using elimination with degree bound \( d^{O(n-s)} \) where \( d = \max_i f_i \).

Then we get that the number of critical points in \( C_\varepsilon \) is at most \( d^{O(n(n-s))} \).
Computing g via Isosingular Deflation

For sufficiently small $\varepsilon > 0$ assume $V_\varepsilon := V(f_1 - a_1\varepsilon, \ldots, f_s - a_s\varepsilon) \subset \mathbb{C}^n$ is smooth and equidimensional of dimension $n - s$. Let $z \in V := \lim_{\varepsilon \to 0} V_\varepsilon \subset \mathbb{C}^n$ generic.

- $F = \{F_1, \ldots, F_N\} \subset \mathbb{R}[x]$ isosingular deflation such that:
  - $\{f_1, \ldots, f_s\} \subseteq F$
  - $F(z) = 0$
  - For $JF(x) \in \mathbb{R}[x]^{N \times n}$ Jacobian of $F$ we have $\text{rank}(JF(z)) = s$

**Proposition (Simplified version) [Harris, Hauenstein, Sz.]

Assume $V$ is irreducible, $z \in V$ generic and $F$ is as above. Let $M(x)$ be a generic linear combination of $s \times s$ submatrices of $JF(x)$. For $g(x) := \det(M(x))$ we have

1. $\text{Sing}(V) \subset V(g)$
2. $\dim(V \cap V(g)) < n - s$.

Furthermore, if the isosingular deflation algorithm takes $k$ iterations then

$$\deg(g) \leq s^{k+1}d$$
Conclusion


Future work:
Try to relax the conditions on $g$ using deformations.
Let $V = \lim_{\varepsilon \to 0} V_\varepsilon$ with $V_\varepsilon$ smooth and equidimensional of dimension $n - s$ for all sufficiently small $\varepsilon > 0$. Assume that $g \in \mathbb{R}[x]$ satisfies

1. $\text{Sing}(V) \subset V(g)$
2. $\dim(V_\varepsilon \cap V(g)) < n - s$ for all sufficiently small $\varepsilon > 0$
   (replacing $\dim(V \cap V(g)) < n - s$).

What are the limits of the real critical points of $g$ on $V_\varepsilon$ in this case?